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Gauge Theory on a Discrete Noncommutative Space

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In this paper, we apply Connes' noncommutative geometry and the Seiberg–Witten map to a discrete noncommutative space consisting of n copies of a given noncommutative space $\mathbf{R}^m$. The explicit action functional of gauge fields on this discrete noncommutative space is obtained.

KEY WORDS: noncommutative geometry; gauge theory; discrete space.

1. INTRODUCTION

Field theories on noncommutative spaces are of great interest now because of the recent development of the superstring theory. It was shown that in the presence of a background Neveu–Schwarz B-field, the gauge theory living on D-branes becomes noncommutative (Connes *et al.*, 1998). On the basis of the existence of the different regularization procedures in string theory, Seiberg and Witten (1999) claimed that certain noncommutative gauge theories are equivalent to commutative ones. In particular, they argued that there exists a map from a commutative gauge field to a noncommutative one, which is compatible with the gauge structure of each. This map has become known as the Seiberg–Witten map.

On the other hand, discrete spaces and corresponding physical theories have been discussed extentively in the literature (see, for example, Bombelli *et al.*, 1987; Feynman, 1982; Finkelstein, 1969; Minsky, 1982; Ruark, 1931; Snyder, 1947; 't Hooft, 1990; Yamamoto, 1984, 1985). In the framework of Connes' noncommutative geometry (Connes, 1985, 1994), finite spaces have been considered to build models in particle physics (Chamseddine *et al.*, 1993; Chamseddine and Connes, 1996, 1997; Connes, 1990, 1996; Connes and Lott, 1990; Coquereaux *et al.*, 1991; Kastler, 1993, 1996; Varilly and Gracia-Bondia, 1993).

Differential calculus and gauge theories on finite sets or finite groups were proposed in the literature (Dimakis and Müller-Hoissen, 1994a,b; Majid, 2000;

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Sitarz, 1992). Especially, the gauge theory on a finite point space was briefly discussed by Cammarata and Coquereaux (1995). In the literature (Hu, 2000; Hu and Sant'Anna, 2002), we give the explicit action functionals of $U(1)$ gauge field on a finite point set and n copies of a connected manifold.

In this paper we first extend the results in the literature (Hu, 2000; Hu and Sant'Anna, 2002) and formulate non-Abelian gauge theory on a finite point space and n copies of a connected manifold. In these two cases, the action functionals are obtained explicitly. We then construct noncommutative gauge theory on the discrete noncommutative space consisting of n copies of a given noncommutative space $\mathbf{R}^m$. The explicit action functional in this case is also obtained.

2. NON-ABELIAN GAUGE THEORY ON *n* COPIES OF A MANIFOLD

2.1. Preliminaries: Differential Calculus on *n*-Point Set

We briefly review the differential calculus on a n-point set. More detailed account of the construction can be found in the literature (Cammarata and Coquereaux, 1995; Dimakis and Müller-Hoissen, 1994a,b; Hu and Sant'Anna, 2002).

Let M be a set of n points $i_1, \ldots, i_n$ $(n < \infty)$, and $\mathcal{A}$ the algebra of complex functions on M with $(fg)(i) = f(i)g(i)$. Let $p_i \in \mathcal{A}$ defined by

$$p_i(j) = \delta_{ij}. \tag{1}$$

It follows that

$$p^* = p, \quad p_i p_j = \delta_{ij} p_j, \quad \sum_i p_i = \mathbf{1}, \tag{2}$$

where $\mathbf{1}(i) = 1$. In other words, p_i is a projector in $\mathcal{A}$. Each $f \in \mathcal{A}$ can be written as

$$f = \sum_i f(i) p_i, \tag{3}$$

where $f(i) \in \mathbf{C}$, a complex number. The algebra $\mathcal{A}$ can be extended to a universal differential algebra $\Omega(\mathcal{A}) = \oplus_{r=0}^{\infty} \Omega^r(\mathcal{A})$ (where $\Omega^0(\mathcal{A}) = \mathcal{A}$) via the action of a linear operator $\mathrm{d} : \Omega^r(\mathcal{A}) \to \Omega^{r+1}(\mathcal{A})$ satisfying

$$d\mathbf{1} = 0, \quad d^2 = 0, \quad d(\omega_r \omega') = (d\omega_r)\omega' + (-1)^r \omega_r \, d\omega',$$

where $\omega_r \in \Omega^r(\mathcal{A})$. The spaces $\Omega^r(\mathcal{A})$ of r-forms are $\mathcal{A}$-bimodules. $\mathbf{1}$ is taken to be the unit in $\Omega(\mathcal{A})$. From the above properties, the set of projectors p_i satisfy the following relations:

$$p_i dp_j = -(dp_i)p_j + \delta_{ij} dp_i, \tag{4}$$

$$\sum_i dp_i = 0. \tag{5}$$

$\Omega(\mathcal{A})$ is an involutive algebra given by

$$(a_0 da_1 \ldots da_n)^* = da_n^* \ldots da_1^* a_0^* \tag{6}$$

where $a_0, a_1, \ldots, a_n \in \mathcal{A}$. We have $\omega^{**} = \omega$ and $(\omega\eta)^* = \eta^*\omega^*$ for $\omega, \eta \in \Omega(\mathcal{A})$. If $\alpha \in \Omega^1$, then $(d\alpha)^* = -d\alpha^*$.

The universal first-order differential calculus Ω^1 is generated by $p_i\, dp_j (i \neq j), i, j = 1, 2, \ldots, n$. Notice that $p_i\, dp_i$ is a linear combination of $p_i\, dp_j (i \neq j)$.

Ω^1 can be defined as the kernel in the algebra $\mathcal{A} \otimes \mathcal{A}$ of the multiplication map. The dimension of Ω^1 is, therefore, $\dim(\mathcal{A} \otimes \mathcal{A}) - \dim \mathcal{A} = n(n-1)$.

Similarly, the compositions of $p_i\, dp_j (i \neq j), i, j = 1, 2, \ldots, n$, generate the higher-order universal differential calculus on M. For example, the universal second-order differential calculus Ω^2 is generated by $p_i\, dp_j\, p_j\, dp_k (i \neq j, j \neq k), i, j, k = 1, 2, \ldots, n$.

Since $\Omega^p = \Omega^1 \otimes_{\mathcal{A}} \ldots \otimes_{\mathcal{A}} \Omega^1$ (p terms), therefore the dimension of Ω^p is $n^p(n-1)^p/n^{p-1} = n(n-1)^p$.

Let $\mathcal{E} = \mathcal{A}^m$ be a free $\mathcal{A}$-module. A connection on $\mathcal{E}$ is a linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ such that

$$\nabla(\Psi a) = \nabla(\Psi)a + \Psi \otimes da, \tag{7}$$

for all $\Psi \in \mathcal{E}, a \in \mathcal{A}$.

Any connection on $\mathcal{E}$ is of the form $\nabla = d + \mathbf{A}$ with $\mathbf{A}^* = -\mathbf{A}$. $\mathbf{A}$ is called a connection 1-form. We can regard $\mathbf{A}$ as an element of $M_m(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$. Here $M_m(\mathcal{A})$ is a $m \times m$ matrix algebra over $\mathcal{A}$. $\mathbf{A}$ can be written as $\mathbf{A} = \sum_{i,j} \mathbf{A}_{ij}\, p_i\, dp_j$ with $\mathbf{A}_{ij} \in M_m(\mathbf{C})$, a $m \times m$ complex matrix, and $\mathbf{A}_{ii} = \mathbf{0}$, a $m \times m$ zero matrix. Especially, $\mathbf{A}^* = -\sum_{i,j} \mathbf{A}_{ji}^*\, p_i dp_j$. From $\mathbf{A}^* = -\mathbf{A}$, we have

$$\mathbf{A}_{ij}^* = \mathbf{A}_{ji}. \tag{8}$$

Let $\mathsf{G} \subset End_{\mathcal{A}}(\mathcal{E}) = M_m(\mathcal{A})$ be a gauge group of $\mathcal{E}$. Then $\mathsf{G} = \sum_i \mathsf{G}_i\, p_i$ with $\mathsf{G}_i \subset M_m(\mathbf{C})$. Notice that $\mathsf{G}_1 = \mathsf{G}_2 = \ldots = \mathsf{G}_n$. There is a natural action of G on the space of connections given by

$$\nabla' = \mathbf{U}\nabla\mathbf{U}^{-1} : \Psi \mapsto \mathbf{U}\nabla(\mathbf{U}^{-1}\Psi), \tag{9}$$

with $\Psi \in \mathcal{E}$ and $\mathbf{U} \in \mathsf{G}$. The connection 1-form $\mathbf{A}$ satisfies

$$\mathbf{A}' = \mathbf{U}\mathbf{A}\mathbf{U}^{-1} + \mathbf{U}d\mathbf{U}^{-1}.$$

Here $\mathbf{U} = \sum_i \mathbf{U}_i\, p_i \in \mathsf{G}$, and $\mathbf{U}_i \in \mathsf{G}_i$.

To make the formulae concise, one introduces

$$\begin{aligned} \mathbf{\Phi} &= \sum_{i,j} \mathbf{\Phi}_{ij}\, p_i \mathrm{d} p_j \\ &= \sum_{i,j} (1 + \mathbf{A}_{ij})\, p_i \mathrm{d} p_i, \end{aligned} \tag{10}$$

with $\mathbf{\Phi}_{ii} = \mathbf{1}$. Here $\mathbf{1}$ is the $m \times m$ identity matrix. One then has

$$\mathbf{\Phi}' = \mathbf{U}\mathbf{\Phi}\mathbf{U}^{-1},$$
$$\mathbf{\Phi}'_{ij} = \mathbf{U}_i \mathbf{\Phi}_{ij} \mathbf{U}_j^{-1}. \tag{11}$$

The curvature of ∇ is defined by $\mathbf{\Theta} = \nabla^2$. It follows that

$$\mathbf{\Theta} = d\mathbf{A} + \mathbf{A}^2. \tag{12}$$

$\mathbf{\Theta}$ transforms in the usual way, $\mathbf{\Theta}' = \mathbf{U}\mathbf{\Theta}\mathbf{U}^{-1}$. From $(d\mathbf{A})^* = -d\mathbf{A}^* = d\mathbf{A}$ and $(\mathbf{A}^2)^* = \mathbf{A}^2$, one has $\mathbf{\Theta}^* = \mathbf{\Theta}$.

As a Matrix valued 2-form, $\mathbf{\Theta}$ can be written as

$$\mathbf{\Theta} = \sum_{i,j,k} \mathbf{\Theta}_{ijk} p_i \, dp_j \, p_j \, dp_k,$$
$$\mathbf{\Theta}_{ijk} = \mathbf{\Phi}_{ij}\mathbf{\Phi}_{jk} - \mathbf{\Phi}_{ik}. \tag{13}$$

2.2. From Fredholm Module to Action Functional on *M*

One of the basic ideas in Connes' noncommutative differential geometry is the Fredholm module (Connes, 1994, and references therein). Applying the Fredholm module to the universal algebra $\Omega(\mathcal{A})$ discussed in the previous subsection, we can obtain an explicit action functional of non-Abelian gauge fields on the finite set M.

Without loss of generality, the Fredholm module $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for the non-Abelian case is the same as the one in the Abelian case (Hu, 2000; Hu and Sant'Anna, 2002). Here $\mathcal{A}$ is the algebra on M defined in the previous section. $\mathcal{H}$ is taken to be a n-dimensional linear space over the complex field C, i.e., $\mathcal{H}$ is just the direct sum $\mathcal{H} = \oplus_{i=1}^{n} \mathcal{H}_i$, $\mathcal{H}_i = \mathrm{C}$. The action of $\mathcal{A}$ on $\mathcal{H}$ is given by

$$\pi(f) = \begin{pmatrix} f(1) & 0 & \dots & 0 \\ 0 & f(2) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & f(n) \end{pmatrix}$$

with $f \in \mathcal{A}$. D is a Hermitian $n \times n$ matrix with $D_{ij} = \overline{D}_{ji}$, and D_{ij} is a linear mapping from $\mathcal{H}_j$ to $\mathcal{H}_i$. The following equality defines an involutive representation of $\Omega(\mathcal{A})$ in $\mathcal{H}$,

$$\pi(da) = [D, \pi(a)], \tag{14}$$

where $a \in \mathcal{A}$. To ensure the differential d satisfies

$$d^2 = 0, \tag{15}$$

one has to impose the following condition on D,

$$D^2 = \mu^2 I, \tag{16}$$

where μ is a real constant and I is the $n \times n$ identity matrix. Notice that the diagonal elements of D commute exactly with the action of $\mathcal{A}$. For the sake of convenience, we can ignore the diagonal elements of D, i. e.,

$$D_{ii} = 0. \tag{17}$$

The projector p_i can be expressed as a $n \times n$ matrix,

$$(\pi(p_i))_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta i}. \tag{18}$$

From (14) and (18), it follows that

$$(\pi(p_i dp_j))_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta j} D_{ij}, \tag{19}$$

$$(\pi(p_i dp_j p_j dp_k))_{\alpha\beta} = \delta_{\alpha i}\delta_{\beta k} D_{ij} D_{jk}. \tag{20}$$

Denote

$$D_{ij}\mathbf{\Phi}_{ij} = \mathbf{H}_{ij}, \tag{21}$$

where $\mathbf{\Phi}_{ij}$ is defined in (10). From the definition of D, we can find that $\mathbf{H}_{ij}$ is a $m \times m$ complex matrix with $\mathbf{H}_{ij}^* = \mathbf{H}_{ji}$. This means that $\mathbf{H} = (\mathbf{H}_{ij}) = \pi(\mathbf{\Phi})$ is a $n \times n$ Hermitian matrix with its elements $m \times m$ submatrices. We have

$$\boldsymbol{H}_{ii} = 0. \tag{22}$$

$\mathbf{H}$ is called the connection matrix on M. From (11) and (21), the transformation rule of $\mathbf{H}_{ij}$ reads

$$\mathbf{H}'_{ij} = \mathbf{U}_i\, \mathbf{H}_{ij}\mathbf{U}_j^{-1}.$$

From (13), (16), and (20)–(22), one has

$$\pi(\mathbf{\Theta}) = \mathbf{H}^2 - \mu^2\mathbf{I}, \tag{23}$$

where $\mathbf{I}$ is the $mn \times mn$ identity matrix.

The transformation rule of $\pi(\mathbf{\Theta}_{ij})$ satisfies

$$\pi(\mathbf{\Theta}'_{ij}) = \mathbf{U}_i \pi(\mathbf{\Theta}_{ij})\mathbf{U}_j^{-1}.$$

$\pi(\mathbf{\Theta})$ is called the curvature matrix on M.

One can define an inner product $\langle | \rangle$ in $\pi(\Omega(\mathcal{A}))$ by setting

$$\langle \alpha | \beta \rangle = \mathrm{Tr}(\alpha^* \beta).$$

Then the action functional of the curvature $\mathbf{\Theta}$ is

$$S = \|\pi(\mathbf{\Theta})\|^2 = \langle \pi(\mathbf{\Theta}) | \pi(\mathbf{\Theta}) \rangle = \mathrm{Tr}(\pi(\mathbf{\Theta})^2. \tag{24}$$

It follows that

$$S = \mathrm{Tr}\mathbf{H}^4 - 2\mu^2 \mathrm{Tr}\mathbf{H}^2 + mn\mu^4. \tag{25}$$

Example. We take $n = 2$ and $\mathsf{G}_1 = \mathsf{G}_2 = U(1)$.

$$\begin{aligned} S &= 2|H_{12}|^4 - 4\mu^2|H_{12}|^2 + 2\mu^4 \\ &= 2\mu^4(|A_{12} + 1|^2 - 1)^2. \end{aligned}$$

This is the Connes' version of Higgs potential.

2.3. Action Functional of Non-Abelian Gauge Fields on *n* Copies of A Manifold

Suppose that each element of a finite set is a manifold, then this finite set forms a disconnected manifold. Let V be an oriented and connected smooth manifold and M, as the previous sections, a n-point set. We see that $V \times M$ is a disconnected manifold consisting of n copies of V.

Let h be a complex function on $V \times M$,

$$h = \sum_i h(i) p_i. \tag{26}$$

Here $h(i)$ is a complex function on V_i, the ith copy of V. The algebra on $V \times M$ is $C^\infty(V) \otimes \mathcal{A}$ with $\mathcal{A}$ the algebra on M.

Denote the differential on M by d_f, i.e., the differential d in Subsections 2 and 3 is replaced by d_f. Let d_s be the usual differential on V, and d the total differential on $V \times M$. It follows that

$$d = d_s + d_f. \tag{27}$$

The nilpotency of d requires that

$$d_s d_f = -d_f d_s. \tag{28}$$

Differentiating (26), we have

$$dh = \sum_i (d_s h(i)) p_i + \sum_i h(i) d_f p_i.$$

A connection 1-form $\mathbf{A}$ on $V \times M$ can be written as

$$\mathbf{A} = \sum_i \mathbf{A}_i p_i + \sum_{i,j} \mathbf{A}_{ij} p_i d_f p_j, \tag{29}$$

$\mathbf{A}$ obeys the usual transformation rule,

$$\mathbf{A}' = \mathbf{U}\mathbf{A}\mathbf{U}^{-1} + \mathbf{U} d \mathbf{U}^{-1}.$$

Here $\mathbf{U} = \sum_i \mathbf{U}_i p_i \in \mathsf{G} \subset M_m(C^\infty(V) \otimes \mathcal{A})$, $\mathbf{U}_i \in \mathsf{G}_i$, and G_i is the gauge group on V_i. Let G be the gauge group on V, then $G = \mathsf{G}_1 = \cdots = \mathsf{G}_n$.

$\mathbf{A}$ has a usual differential degree and a finite-difference degree (α, β) adding up to 1:

$$\mathbf{A}^{(1,0)} = \sum_i \mathbf{A}_i p_i. \tag{30}$$

It is the continuous part of $\mathbf{A}$. $\mathbf{A}_i$ is a Lie algebra valued 1-form on V_i and $\mathbf{A}_i^* = -\mathbf{A}_i$.

$$\mathbf{A}^{(0,1)} = \sum_{i,j} \mathbf{A}_{ij} p_i d_f p_j. \tag{31}$$

It is the connection 1-form on M, and is well studied in the previous subsection.

The curvature of $\mathbf{A}$ is given by

$$\mathbf{\Theta} = d\mathbf{A} + \mathbf{A}^2.$$

It can be seen that $\mathbf{\Theta}$ transforms in the usual way, $\mathbf{\Theta}' = \mathbf{U}\mathbf{\Theta}\mathbf{U}^{-1}$. As a 2-form, $\mathbf{\Theta}$ can be written as

$$\begin{aligned}\mathbf{\Theta} = {} & \sum_i (d_s \mathbf{A}_i + \mathbf{A}_i \wedge \mathbf{A}_i) p_i + \sum_{i,j} (d_s \mathbf{\Phi}_{ij} + \mathbf{A}_i \mathbf{\Phi}_{ij} - \mathbf{\Phi}_{ij} \mathbf{A}_j) p_i d_f p_j \\ & + \sum_{i,j,k} \mathbf{\Theta}_{ijk} p_i d_f p_j p_j d_f p_k,\end{aligned} \tag{32}$$

$$\mathbf{\Theta}_{ijk} = \mathbf{\Phi}_{ij} \mathbf{\Phi}_{jk} - \mathbf{\Phi}_{ik}. \tag{33}$$

Applying the Fredholm module to the above formula, we have

$$\pi(\mathbf{\Theta}_{ij}) = \mathbf{F}_i \delta_{ij} + d_s \mathbf{H}_{ij} + \mathbf{A}_i \mathbf{H}_{ij} - \mathbf{H}_{ij} \mathbf{A}_j + (\mathbf{H}^2 - \mu^2 \mathbf{I})_{ij}. \tag{34}$$

Here $\mathbf{F}_i$ is the curvature of $\mathbf{A}_i$, $\mathbf{F}_i = d_s \mathbf{A}_i + \mathbf{A}_i \wedge \mathbf{A}_i$.

We see that $\pi(\mathbf{\Theta})$ has a usual differential degree and a finite-difference degree (α, β) adding up to 2. Let us begin with the term in $\pi(\mathbf{\Theta})$ of bidegree (2, 0):

$$\mathbf{\Theta}_{ij}^{(2,0)} = \mathbf{F}_i \delta_{ij}, \tag{35}$$

i.e., $\mathbf{\Theta}_{ii}^{(2,0)} = \mathbf{F}_i$, and $\mathbf{\Theta}_{ij}^{(2,0)} = \mathbf{0} (i \neq j)$.

$\mathbf{\Theta}_{ii}^{(2,0)}$ obeys the transformation rule,

$$\mathbf{\Theta}_{ii}^{\prime(2,0)} = \mathbf{U}_i \mathbf{\Theta}_{ii}^{(2,0)} \mathbf{U}_i^{-1}.$$

$\mathbf{\Theta}^{(2,0)}$ is the continuous part of the field strength.

Next, we look at the component $\mathbf{\Theta}^{(1,1)}$ of bidegree (1, 1):

$$\mathbf{\Theta}_{ij}^{(1,1)} = d_s \mathbf{H}_{ij} + \mathbf{A}_i \mathbf{H}_{ij} - \mathbf{H}_{ij} \mathbf{A}_j. \tag{36}$$

One can find that $\mathbf{\Theta}_{ij}^{(1,1)}$ transforms as the following:

$$\mathbf{\Theta}_{ij}^{\prime(1,1)} = \mathbf{U}_i \mathbf{\Theta}_{ij}^{(1,1)} \mathbf{U}_j^{-1}.$$

$\mathbf{\Theta}^{(1,1)}$ corresponds to the interaction between V and M.

We can define a covariant derivative of $\mathbf{H}_{ij}$ as

$$\mathrm{D}_\mu \mathbf{H}_{ij} = \partial_\mu \mathbf{H}_{ij} + \mathbf{A}_{i\mu} \mathbf{H}_{ij} - \mathbf{H}_{ij} \mathbf{A}_{j\mu}. \tag{37}$$

Therefore $\mathbf{\Theta}_{ij}^{(1,1)} = \mathrm{D}_\mu \mathbf{H}_{ij} dx^\mu$. From now on, the Einstein sum convention for the indices μ and ν is adopted.

Finally, we have the component $\mathbf{\Theta}^{(0,2)}$ of degree (0, 2):

$$\mathbf{\Theta}^{(0,2)} = \mathbf{H}^2 - \mu^2 \mathbf{I}, \tag{38}$$

with

$$\mathbf{\Theta}_{ij}^{\prime(0,2)} = \mathbf{U}_i \mathbf{\Theta}_{ij}^{(0,2)} \mathbf{U}_j^{-1}.$$

$\mathbf{\Theta}^{(0,2)}$ corresponds to the field strength on the finite set M.

We then obtain the action functional on $V \times M$:

$$S = \int_V \mathcal{L} d\nu$$

The Lagrangian density is given by the following formulae:

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_1 + \mathcal{L}_0, \tag{39}$$

$$\mathcal{L}_2 = -\frac{1}{2} \sum_i \mathrm{Tr}(\mathbf{F}_{i\mu\nu} \mathbf{F}_i^{\mu\nu}), \tag{40}$$

$$\begin{aligned} \mathcal{L}_1 &= \mathrm{Tr}[(\mathrm{D}_\mu \mathbf{H})(\mathrm{D}^\mu \mathbf{H})^*] = \sum_{i,j} \mathrm{Tr}[(\mathrm{D}_\mu \mathbf{H}_{ij})(\mathrm{D}^\mu \mathbf{H}_{ij})^*] \\ &= \sum_{i,j} \mathrm{Tr}\big[(\partial_\mu \mathbf{H}_{ij} + \mathbf{A}_{i\mu} \mathbf{H}_{ij} - \mathbf{H}_{ij} \mathbf{A}_{j\mu})\big(\partial^\mu \mathbf{H}_{ji} + \mathbf{A}_j^\mu \mathbf{H}_{ji} - \mathbf{H}_{ji} \mathbf{A}_i^\mu\big)\big] \end{aligned} \tag{41}$$

$$\mathcal{L}_0 = \mathrm{Tr}\mathbf{H}^4 - 2\mu^2 \mathrm{Tr}\mathbf{H}^2 + mn\mu^4. \tag{42}$$

The term $\mathcal{L}_2$ is the usual term describing the Lagrange for a $G \times G \times \cdots \times G$ (n terms) connection. $\mathbf{H}_{ij}(i \neq j, i, j = 1, 2, \ldots, n)$ in $\mathcal{L}_1$ give a mass to the gauge fields $\mathbf{A}_i$ and $\mathbf{A}_j$.

Example. We take $G = U(1)$. The Lagrangian density of $U(1)$ gauge fields on $V \times M$ is given by the following formulae (Hu, 2000; Hu and Sant'Anna, 2002):

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_1 + \mathcal{L}_0, \tag{43}$$

$$\mathcal{L}_2 = -\frac{1}{4} \sum_i F_{i\mu\nu} F_i^{\mu\nu}, \tag{44}$$

where $F_{i\mu\nu} = \partial_\mu A_{i\nu} - \partial_\nu A_{i\mu}$.

$$\mathcal{L}_1 = \mathrm{Tr}\big[(\mathrm{D}_\mu H)(\mathrm{D}^\mu H)^*\big] = \sum_{i,j}\big[(\mathrm{D}_\mu H_{ij})(\mathrm{D}^\mu H_{ij})^*\big]$$
$$= \sum_{i,j}[(\partial_\mu + A_{i\mu} - A_{j\mu})H_{ij}]\big[\big(\partial^\mu + A_j^\mu - A_i^\mu\big)H_{ji}\big], \tag{45}$$
$$\mathcal{L}_0 = \mathrm{Tr}H^4 - 2\mu^2\mathrm{Tr}\mathrm{H}^2 + n\mu^4.$$
$$= \sum_{i,j,k,l}\left(\sum_\alpha e_i^\alpha e_j^\alpha e_k^\alpha e_l^\alpha\right)\phi_i\phi_j\phi_k\phi_l - \frac{2n}{n-1}\mu^2\left(\sum_i \phi_i^2\right) + n\mu^4. \tag{46}$$

Here $H = (H_{ij})$ is a $n \times n$ matrix. $\phi_i(i = 1, \ldots, n-1)$ is a real parameter field.

3. GAUGE THEORY ON *n* COPIES OF NONCOMMUTATIVE SPACE $\mathbf{R}^n$

3.1. Moyal Star Product

We consider a noncommutative Eucleadian space $\mathbf{R}^m$ with coordinates $\hat{x}^i$ characterized by the algebra

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}, \tag{47}$$

where θ^{ij} is an antisymmetric constant tensor with $\theta^{ij} = -\theta^{ji}$. Field theories in such a space can be realized as a deformation of the usual field theory in an ordinary (commutative) space by changing the product of two fields to the Moyal star product (Mayal, 1949) defined by

$$f(x) * g(x) = \exp\left(\frac{i}{2}\theta^{kl}\frac{\partial}{\partial y^k}\frac{\partial}{\partial z^l}\right) f(y)g(z)|_{y=z=x}. \tag{48}$$

Note that the first term on the right side gives the ordinary product. Also the commutator (47) is realized as

$$[x^i, x^j]_* \equiv x^i * x^j - x^j * x^i = i\theta^{ij}. \tag{49}$$

3.2. Seiberg–Witten Map

Let $\hat{\mathbf{A}}_i$ be a noncommutative gauge fields on a noncommutative Euclidean space $\mathbf{R}^m$, whose coordinates obey $[x^i, x^j]_* = i\theta^{ij}$. Denote $\mathbf{A}_i$ the counterpart of $\hat{\mathbf{A}}_i$, the ordinary gauge field on the ordinary Euclidean space R^m. The map between $\mathbf{A}_i$ and $\hat{\mathbf{A}}_i$, called the Seiberg-Witten map (Seiberg and Witten, 1999), is characterized by the differential equation with respect to θ,

$$\delta\hat{\mathbf{A}}_i(\theta) = -\frac{1}{4}\delta\theta^{jk}[\hat{\mathbf{A}}_j * (\partial_k\hat{\mathbf{A}}_i + \hat{\mathbf{F}}_{ki}) + (\partial_k\hat{\mathbf{A}}_i + \hat{\mathbf{F}}_{ki}) * \hat{\mathbf{A}}_j], \tag{50}$$

with the initial condition,

$$\hat{A}_i(\theta = 0) = \mathbf{A}_i, \tag{51}$$

Here $*$ is the Moyal star product. The field strength $\hat{\mathbf{F}}_{ij}$ is defined as

$$\hat{\mathbf{F}}_{ij} = \partial_i \hat{\mathbf{A}}_j - \partial_j \hat{\mathbf{A}}_i - i\hat{\mathbf{A}}_i * \hat{\mathbf{A}}_j + i\hat{\mathbf{A}}_j * \hat{\mathbf{A}}_i. \tag{52}$$

The differential Eq. (50) is known as the Seiberg-Witten equation.

3.3. Gauge Theory on *n* Copies of Noncommutative Space $\mathbf{R}^m$

Naively, to get a physical quantity on a noncommutative space, we simply take this quantity on the corresponding commutative space and replace all products by the $*$ products.

Let $\mathbf{R}^m$ be a noncommutative Euclidean space and M a n-point set. Then $\mathbf{R}^m \times M$ is a discrete noncommutative space consisting of n copies of $\mathbf{R}^m$. Now we construct noncommutative gauge theory on $\mathbf{R}^m \times M$.

A noncommutative connection 1-form $\hat{\mathbf{A}}$ on $\mathbf{R}^m \times M$ can be written as

$$\hat{\mathbf{A}} = \sum_i \hat{\mathbf{A}}_i p_i + \sum_{i,j} \hat{\mathbf{A}}_{ij} p_i d_f p_j, \tag{53}$$

$\hat{A}$ has a differential degree (α, β) adding up to 1:

$$\hat{\mathbf{A}}^{(1,0)} = \sum_i \hat{\mathbf{A}}_i p_i. \tag{54}$$

$$\hat{\mathbf{A}}^{(0,1)} = \sum_{i,j} \hat{\mathbf{A}}_{ij} p_i d_f p_j. \tag{55}$$

The curvature of $\hat{\mathbf{A}}$ is given by

$$\hat{\boldsymbol{\Theta}} = d\hat{\mathbf{A}} + \hat{\mathbf{A}} * \hat{\mathbf{A}}.$$

$\boldsymbol{\Theta}$ can be written as

$$\hat{\boldsymbol{\Theta}} = \sum_i (d_s\hat{\mathbf{A}}_i + \hat{\mathbf{A}}_i * \wedge \hat{\mathbf{A}}_i) p_i + \sum_{i,j} (d_s\hat{\boldsymbol{\Phi}}_{ij} + \hat{\mathbf{A}}_i * \boldsymbol{\Phi}_{ij} - \hat{\boldsymbol{\Phi}}_{ij} * \hat{\mathbf{A}}_j) p_i d_f p_j$$
$$+ \sum_{i,j,k} \hat{\boldsymbol{\Theta}}_{ijk} p_i d_f p_j p_j d_f p_k, \tag{56}$$

$$\hat{\boldsymbol{\Theta}}_{ijk} = \hat{\boldsymbol{\Phi}}_{ij} * \hat{\boldsymbol{\Phi}}_{jk} - \hat{\boldsymbol{\Phi}}_{ik}. \tag{57}$$

Applying the Fredholm module to the above formula, we have

$$\pi(\hat{\boldsymbol{\Theta}}_{ij}) = \hat{\mathbf{F}}_i \delta_{ij} + d_s \hat{\mathbf{H}}_{ij} + \hat{\mathbf{A}}_i * \mathbf{H}_{ij} - \hat{\mathbf{H}}_{ij} * \hat{\mathbf{A}}_j + (\hat{\mathbf{H}} * \hat{\mathbf{H}} - \mu^2 \mathbf{I})_{ij}. \tag{58}$$

Here $\hat{\mathbf{F}}_i$ is the curvature of $\hat{\mathbf{A}}_i$, $\hat{\mathbf{F}}_i = d_s\hat{\mathbf{A}}_i + \hat{\mathbf{A}}_i * \wedge \hat{A}_i$.

We see that $\pi(\hat{\boldsymbol{\Theta}})$ has a differential degree (α, β) adding up to 2. Let us begin with the term in $\pi(\hat{\boldsymbol{\Theta}})$ of bidegree (2, 0):

$$\hat{\boldsymbol{\Theta}}^{(2,0)}_{ij} = \hat{\mathbf{F}}_i \delta_{ij}, \tag{59}$$

i.e., $\hat{\boldsymbol{\Theta}}^{(2,0)}_{ii} = \hat{\mathbf{F}}_i$, and $\hat{\boldsymbol{\Theta}}^{(2,0)}_{ij} = \mathbf{0}(i \neq j)$.

Next, we look at the component $\hat{\boldsymbol{\Theta}}^{(1,1)}$ of bidegree (1,1):

$$\hat{\boldsymbol{\Theta}}^{(1,1)}_{ij} = d_s \hat{\mathbf{H}}_{ij} + \hat{\mathbf{A}}_i * \hat{\mathbf{H}}_{ij} - \hat{\mathbf{H}}_{ij} * \hat{\mathbf{A}}_j. \tag{60}$$

$\hat{\boldsymbol{\Theta}}^{(1,1)}$ corresponds to the interaction between $\mathbf{R}^m$ and M.

We can define a covariant derivative of $\hat{\mathbf{H}}_{ij}$ as

$$D_\mu * \hat{\mathbf{H}}_{ij} = \partial_\mu \hat{\mathbf{H}}_{ij} + \hat{\mathbf{A}}_{i\mu} * \hat{\mathbf{H}}_{ij} - \hat{\mathbf{H}}_{ij} * \hat{\mathbf{A}}_{j\mu}. \tag{61}$$

Therefore $\hat{\boldsymbol{\Theta}}^{(1,1)}_{ij} = D_\mu * \hat{\mathbf{H}}_{ij} * dx^\mu$. Here the Einstein sum convention for the indices μ and ν is adopted.

Finally, we have the component $\hat{\boldsymbol{\Theta}}^{(0,2)}$ of degree (0, 2):

$$\hat{\boldsymbol{\Theta}}^{(0,2)} = \hat{\mathbf{H}} * \hat{\mathbf{H}} - \mu^2 \mathbf{I}, \tag{62}$$

The Lagrangian density is given by the following formulae:

$$\hat{\mathcal{L}} = \hat{\mathcal{L}}_2 + \hat{\mathcal{L}}_1 + \hat{\mathcal{L}}_0, \tag{63}$$

$$\hat{\mathcal{L}}_2 = -\frac{1}{2} \sum_i \mathrm{Tr}\big(\hat{\mathbf{F}}_{i\mu\nu} * \hat{\mathbf{F}}_i^{\mu\nu}\big), \tag{64}$$

$$\begin{aligned}\hat{\mathcal{L}}_1 &= \mathrm{Tr}[(\mathrm{D}_\mu * \hat{\mathbf{H}}) * (\mathrm{D}^\mu * \hat{\mathbf{H}})^*] = \sum_{i,j} \mathrm{Tr}[(\mathrm{D}_\mu * \hat{\mathbf{H}}_{ij}) * (\mathrm{D}^\mu * \hat{\boldsymbol{H}}_{ij})^*] \\ &= \sum_{i,j} \mathrm{Tr}[(\partial_\mu \hat{\mathbf{H}}_{ij} + \hat{\mathbf{A}}_{i\mu} * \hat{\mathbf{H}}_{ij} - \hat{\mathbf{H}}_{ij} * \hat{\mathbf{A}}_{j\mu}) \\ &\quad * (\partial^\mu \hat{\mathbf{H}}_{ji} + \hat{\mathbf{A}}_j^\mu * \hat{\mathbf{H}}_{ji} - \hat{\mathbf{H}}_{ji} * \hat{\mathbf{A}}_i^\mu)],\end{aligned} \tag{65}$$

$$\hat{\mathcal{L}}_0 = \mathrm{Tr}\hat{\mathbf{H}} * \hat{\mathbf{H}} * \hat{\mathbf{H}} * \hat{\mathbf{H}} - 2\mu^2 \mathrm{Tr}\hat{\mathbf{H}} * \hat{\mathbf{H}} + mn\mu^4. \tag{66}$$

Example. The Lagrangian density of the noncommutative $U(1)$ gauge fields on noncommutative $\mathbf{R}^m \times M$ is given by the following formulae:

$$\hat{\mathcal{L}}_2 = -\frac{1}{4} \sum_i \hat{F}_{i\mu\nu} * \hat{F}_i^{\mu\nu}, \tag{67}$$

$$\begin{aligned}\hat{\mathcal{L}}_1 &= \mathrm{Tr}[(\mathrm{D}_\mu * \hat{H}) * (\mathrm{D}^\mu * \hat{H})^*] = \sum_{i,j} [(\mathrm{D}_\mu * \hat{H}_{ij}) * (\mathrm{D}^\mu * \hat{H}_{ij})^*] \\ &= \sum_{ij} [(\partial_\mu + \hat{A}_{i\mu} - \hat{A}_{j\mu}) * \hat{H}_{ij}] * [(\partial^\mu + \hat{A}_j^\mu - \hat{A}_i^\mu) * \hat{H}_{ji}],\end{aligned} \tag{68}$$

$$\hat{\mathcal{L}}_0 = \mathrm{Tr}\hat{H} * \hat{H} * \hat{H} * \hat{H} - 2\mu^2 \mathrm{Tr}\hat{H} * \hat{H} + n\mu^4.$$
$$= \sum_{i,j,k,l} \left(\sum_{\alpha} e_i^{\alpha} e_j^{\alpha} e_k^{\alpha} e_l^{\alpha} \right) \hat{\phi}_i * \hat{\phi}_j * \hat{\phi}_k * \hat{\phi}_l$$
$$- \frac{2n}{n-1}\mu^2 \left(\sum_i \hat{\phi}_i * \hat{\phi}_i \right) + n\mu^4. \qquad (69)$$

Since the Moyal star product of two fields differs from the ordinary one only by a total divergence, the quadratic term $\hat{\phi}_i * \hat{\phi}_i$ in (69) can be replaced by $\hat{\phi}_i^2$.

We now consider the simpliest case, i.e.,

$$\hat{A}_i = \hat{A}, \qquad i = 1, 2, \ldots, n. \qquad (70)$$

Here $\hat{A}$ is a $U(1)$ gauge field on noncommutative $\mathbf{R}^m$. The physical meaning of the above asumptions is: there exists unique noncommutative gauge field, i.e., the noncommutative Maxwell electromagnetic field over all copies of $\mathbf{R}^m$.

In this special case, the Lagrangian density is given by the following formulae:

$$\hat{\mathcal{L}}_2 = -\frac{1}{4} n(\hat{F}_{\mu\nu} * \hat{F}^{\mu\nu}), \qquad (71)$$

$$\hat{\mathcal{L}}_1 = \frac{n}{n-1} \sum_i (\partial_\mu \hat{\phi}_i) * (\partial^\mu \hat{\phi}_i), \qquad (72)$$

and $\hat{\mathcal{L}}_0$ is the same as the formula (69). Notice that the term $(\partial_\mu \hat{\phi}_i) * (\partial^\mu \hat{\phi}_i)$ in (72) can be replaced by $(\partial_\mu \hat{\phi}_i)(\partial^\mu \hat{\phi}_i)$ and the Lagrangian density $\hat{\mathcal{L}}$ corresponds to the Landau–Ginsburg model in the noncommutative $\mathbf{R}^m$ (see for example, Aref'eva *et al.*, 2000; Gubser and Sondhi, 2001).

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